MATHEMATICS

Mob. : 9470844028
9546359990

AIM POINT
MATHEMATICS
DIR. FIROZ AHMAD
M.Sc. (Maths), B.Ed, M.Phil (Maths)

RAM RAJYA MORE, SIWAN

XIth, XIIth, TARGET IIT-JEE (MAIN + ADVANCE) & COMPATETIVE EXAM FOR XII (PQRS)

INDEFINITE INTERATION

& Their Properties

CONTENTS

THINGS TO REMEMBER

✷ Introducation

Integration is called the inverse process of differentiation. This helps to find the function whose differential coefficient is known.

Thus, a function $f(x)$ is called a primitive or an anti-derivative of a function $f(x)$, if

$$F'(x) = f(x)$$

Since, the differential coefficient of a constant is zero.

$$\frac{d}{dx}[F(x) + c] = \frac{d}{dx} F(x) + 0 = f(x)$$

Hence, $$\int f(x)dx = F(x) + c$$

where, ∫ is the notation of integration, $f(x)$ is called the integrand and x is called the variable of integration. c is the constant of integeration and can take any constant value. This shows that $F(x)$ and $F(x) + c$ are both integrals of the same function $f(x)$. Thus, for different value of c, we obtain different integral of $f(x)$. This implies that the integral of $f(x)$ is not definite. Due to this $F(x)$ is called the indefinite integral of $f(x)$.

✷ Standard Formula

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c, \ (n \neq -1)$

2. $\int \frac{dx}{x} = \log_e |x| + c$

3. $\int e^x dx = e^x + c$

4. $\int a^x dx = \frac{a^x}{\log_e a} + c$

5. $\int \sin x \, dx = -\cos x + c$

6. $\int \cos x \, dx = \sin x + c$

7. $\int \sec^2 x \, dx = \tan x + c$

8. $\int \mathrm{cosec}^2 x \, dx = -\cot x + c$

9. $\int \sec x \tan x \, dx = \sec x + c$

10. $\int \mathrm{cosec}\, x \cot x \, dx = -\mathrm{cosec}\, x + c$

11. $\int \cot x \, dx = \log |\sin x| + c$

12. $\int \tan x \, dx = -\log |\cos x| + c$

13. $\int \sec x \, dx = \log |\sec x + \tan x| + c = \log \tan \left|\frac{\pi}{4} + \frac{x}{2}\right| + c$

14. $\int \mathrm{cosec}\, x \, dx = \log |\mathrm{cosec}\, x - \cot x| + c = \log \tan \left(\frac{x}{2}\right) + c$

15. $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right)+c$

16. $\int \dfrac{dx}{x^2+a^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}+c$

17. $\int \dfrac{dx}{x\sqrt{x^2-a^2}} = \dfrac{1}{a}\sec^{-1}\dfrac{x}{a}+c$

18. $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right|+c,\ x>a$

19. $\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\log\left|\dfrac{a+x}{a-x}\right|+c$

20. $\int \dfrac{dx}{\sqrt{a^2+x^2}} = \log|x+\sqrt{a^2+x^2}|+c = \sinh^{-1}\left(\dfrac{x}{a}\right)+c$

21. $\dfrac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}|+c = \cosh^{-1}\left(\dfrac{x}{a}\right)+c$

22. $\int \sqrt{a^2-x^2}\,dx = \dfrac{1}{2}x\sqrt{a^2-x^2}+\dfrac{1}{2}a^2\sin^{-1}\left(\dfrac{x}{a}\right)+c$

23. $\int \sqrt{a^2+x^2}\,dx = \dfrac{1}{2}x\sqrt{a^2+x^2}+\dfrac{1}{2}a^2\log|x+\sqrt{a^2+x^2}|+c$

24. $\int \sqrt{x^2-a^2}\,dx = \dfrac{1}{2}x\sqrt{x^2-a^2}-\dfrac{1}{2}a^2\log|x-\sqrt{x^2-a^2}|+c$

25. $\int e^{ax+b}\,[af(x)+f(x)]+dx = e^{ax+b}f(x)+c$

26. $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+c$

27. $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+c = \dfrac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\dfrac{b}{a}\right)+c$

✷ Rules for Integration

1. $\int \{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)\,dx$
2. $\int k\,f(x)dx = k\int f(x)dx$, where k is a contant.
3. $\dfrac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$
4. $\int f(x)\,dx = g(x)+c$

$\Rightarrow \quad \int f(ax+b)\,dx = \dfrac{g(ax+b)}{a}+c$

✷ Integration by Substitution

Integration of certain function cannot be abtained directly, if they are not in one of the standard forms, but they may be reduced to standard forms by proper substitution.

If g(x) is a continuous diffecentible function, then to evaluate integrals of the form

$$I = \int f[g(x)].\ g'(x)\ dx$$

We substitute

$$g(x) = t \quad \text{and} \quad g'(x)dx = dt$$

The substitution reduces the integral to $\int f(t)dt$. After evaluating this integral we substitute back the value of t.

Function	Substitution
$f\left(\sqrt{a^2 - x^2}\right)$	$x = a \sin\theta$ or $x = a \cos\theta$
$f\left(\sqrt{x^2 - a^2}\right)$	$x = a \sec\theta$ or $x = a\ \text{cosec}\ \theta$
$f(x^2 + a^2)$, $f\left(\sqrt{x^2 + a^2}\right)$	$x = a \tan\theta$ or $x = a \cot\theta$
$f\left(\sqrt{\frac{a-x}{a+x}}\right)$, $f\left(\sqrt{\frac{a+x}{a-x}}\right)$	$x = a \cos 2\theta$
$f\left(\sqrt{\frac{x-a}{b-x}}\right)$, $f\left(\sqrt{(x-a)(x-b)}\right)$	$x = a\cos^2\theta + b\sin^2\theta$

✷ Integration of Different Types of Function

1. Integration of type $\sin^m x \cos^n x\ dx$

   (a) If m is an odd integer, put $\cos x = t$

   (b) If n is an odd integer, put $\sin x = t$

   (c) If m + n is negative even integer, then put $\tan x = t$ or $\cot x = t$.

   (d) If m and are even natural number, then converts higher power into higher angles.

2. If the integrals are of the form $\int \frac{dx}{ax^2+bx+c}$, $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ and $\int \sqrt{ax^2+bx+x}\ dx$, the express

   ax^2 + bx + c as the sum of difference of two squares ie, in the form of perfect square and then, apply the standard results.

3. If the integrals are of the form $\int \frac{px+q}{ax^2+bx+c}$ dx, $\int \frac{px+q}{\sqrt{ax^2+bx+c}}$ dx and $\int (px+q)\sqrt{ax^2+bx+x}\ dx$.

   Then put $px + q$ = A (differential coefficient of ax^2 + bx + c) + B. Find A and B by comparing the coefficients of x and constant term on bothe the side of equation. Then $\int \frac{px+q}{ax^2+bx+c}$ dx =

A $\int \frac{ax+b}{ax^2+bx+c}$ + A $\int \frac{1}{ax^2+bx+c}$. Now, we can integrates it easily. Similarly, other tow cases.

4. If the integrals are of the form $\int \frac{dx}{(ax+b)\sqrt{lx+m}}$ and $\int \frac{dx}{(ax^2+bx+c)\sqrt{lx+m}}$, then put $lx + m = t^2$.

5. If the integral is of the form $\int \frac{dx}{(ax+b)\sqrt{lx^2+mx+n}}$ then put ax + b = $\frac{1}{t}$ and for $\int \frac{dx}{(ax^2+b)\sqrt{lx^2+m}}$, put $x = \frac{1}{t}$.

6. If the integral is of the form $\int \frac{dx}{(ax^2+bx+x)\sqrt{lx^2+mx+n}}$ then put $\frac{ax^2+bx+c}{lx^2+mx+n} = t^2$.

7. If integral is of the form $\int \frac{x^2+1}{x^2+kx^2+1}$ dx, where k is any constant, then divide numerator (Nr) and denominator (Dr) by x^2 and $x \mp \frac{1}{x}$ = t.

8. If the integral is of the form $\int \frac{dx}{(ax+b)^m(cx+d)^n}$, where m and n are positive integers, then put $\frac{ax+b}{cx+d} = t$.

9. If the intergral are of the form $\int \frac{dx}{a\cos^2 x+b}$, $\int \frac{dx}{a+b\sin^2 x}$ and $\int \frac{dx}{a^2\sin x+2b\sin x\cos x+c^2\cos^2 x}$, then multiply the Nr or Dr by $\sec^2 x$ and then. put tan x = t. (use $\sec^2 x = 1 + \tan^2 x$).

10. If the integrals are of the form $\int \frac{1}{a\cos x+b\sin x}dx$, $\int \frac{dx}{a+b\cos x}$ and $\int \frac{dx}{a+\sin x}$ and then convert sines and cosines into their respective tangents of half the angle and then, put $\tan\frac{x}{2} = t$.

11. If the integrals are of the form $\int \frac{p\cos x+q\sin x+r}{a\cos x+b\sin x+n}dx$, $\int \frac{p\cos x+q\sin x}{a\cos x+b\sin x}dx$, $\int \frac{p\cos x}{a\cos x+b\sin x}dx$, and $\int \frac{q\sin x}{a\cos x+b\sin x}dx$ then express the numerator as

$$\text{Nr} \equiv \text{A(Dr)} + \text{B}\frac{d}{dx}\text{(Dr)} + \text{c}$$

[The value of c is zero in last three forms] and compare the coefficients of sin x and cos x and then, proceed.

12. If integral is of the form $\int \frac{ae^x + be^{-x}}{ce^x + de^{-x}} dx$ then put $ae^x + be^{-x} = A(ce^x + de^{-x}) + B(ce^x - de^{-x})$

Now, find $\int \frac{ae^x + be^{-x}}{ce^x + de^{-x}} dx = A\,dx + B\frac{f'(x)}{f(x)}dx$, where $f(x) = ce^x + de^{-x}$.

✷ Integration by Parts

If u and v are the differentiable function fo x, then

$$\int u \cdot v\, dx = u \int v\, dx - \int\left[\left(\frac{d}{dx}(u)\right)\left(\int v dx\right)\right]dx .$$

ie, the integral of the product of two fucntions = (first functions) x (Integral of the second function) – Integral of {(differentiation of first function) x (Indtegral of second function)}.

How to Choose Ist and IInd Function

If two function are of different types take the function as Ist which comes first in the word **ILATE**, where I stands for inverse cirular functio, L stands for logarithmic fucntion, A stands for algebraic function, T stands for tirgonometric and E for exponential function.

✷ Some Important Points

1. If $f(x)$ and $g(x)$ are same functions, then $\{f(x) - g(x)\}$ dx is constant.
2. As integration and differentiation are inverse process, many times result of integration can be verified by differentiating its perimitive of anti-derivative.

   eg, $$\frac{\log(x+1) - \log x}{x(x+1)} dx = -\frac{1}{2}\left(\log\left(\frac{x+1}{x}\right)\right)^2 + c$$

   can be verified by differentiating its perimitive.

   $$\Rightarrow \quad \frac{d}{dx}\left(-\frac{1}{2}\log\left(\frac{x+1}{x}\right)^2\right) = -\frac{1}{2}(2)\left(\log\left(\frac{x+1}{x}\right)\right)\left(\frac{x+1}{x}\right)\left(\frac{x-x-1}{x}\right)$$

   $$= \frac{\log(x+1) - \log x}{x(x+1)}$$

Some Integrals which cannot be Found

Any function continuous on an interval (a, b) has an anti-derivative n that interval In other words, there exists a function F(x) such F′(x) = f(x).

However, not every anti-derivative F(x), even when it exists, is expressibe in closed form in terms of elementary functions such as polynomials, trigonometric, logarithmic, exponential functions etc. Then, we say that such anti-derivatives of integrals "cannot be fountd".

Note :

- If the integral contains a single logarithmic or single inverse trigonometric function take unity as the second function.
- If the integrals of both the functions are known, the function which is easy to integrate is taken as the secon function.
- In certain cases integration by parts will lead to a simple equation involving the integral. Solve the equation and determine the integral.